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Finite-size corrections to Poisson approximations in general renewal-success processes

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Abstract

Consider a renewal process, and let $K \geqslant 0$ denote the random duration of a typical renewal cycle. Assume that on any renewal cycle, a rare event called "success" can occur. Such successes lend themselves naturally to approximation by Poisson point processes. If each success occurs after a random delay, however, Poisson convergence can be relatively slow, because each success corresponds to a time interval, not a point. If K is an arithmetic variable, a "finite-size correction" (FSC) is known to speed Poisson convergence by providing a second, subdominant term in the appropriate asymptotic expansion. This paper generalizes the FSC from arithmetic K to general K. Genomics applications require this generalization, because they have already heuristically applied the FSC to p-values involving absolutely continuous distributions. The FSC also sharpens certain results in queuing theory, insurance risk, traffic flow, and reliability theory.

Keywords: Renewal-success process; Finite-size correction; Regenerative process with a rare event

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1. Introduction

With appropriate subscripts in the following, K and T denote non-negative real random variables, and $\mathcal{E}$ denotes an event. As is usual, the indicator $\mathbb{I}(\mathcal{E})$ of an event $\mathcal{E}$ equals 1 if $\mathcal{E}$ occurs, and 0 if it does not. For convenience in the following, define$K_{-1} := 0$. (The symbol ":=" or "=:" denotes a definition of the left or right side of the equality.) A sequence of random triplets $(K_i, T_i, \mathbb{I}(\mathcal{E}_i))$ $(i = 0, 1, 2, \ldots)$ with $\mathbb{P}(K_i > K_{i-1}) > 0$ and $K_{i-1} \leqslant T_i \leqslant K_i$ constitutes a renewal-success process, if two conditions hold. First, the "cycle triplets" $(K_i - K_{i-1}, T_i - K_{i-1}, \mathbb{I}(\mathcal{E}_i))$ are independent of each other. (Within each cycle triplet, the coordinates are permitted to be dependent on each other.) Second, the cycle triplets must be identically distributed, except possibly the 0th triplet $(K_0, T_0, \mathbb{I}(\mathcal{E}_0))$. (The exception permits the possibility of delayed renewals.)

Intuitively, $\{K_i\}$ is a renewal process whose ith renewal cycle is the time interval $(K_{i-1}, K_i]$. Within the ith cycle, an event $\mathcal{E}_i$ might occur. The event $\mathcal{E}_i$ is called "success"; its complement $\mathcal{E}_i^c$ is called "failure." If success occurs, it occurs at a random success time T_i within the ith cycle. If failure occurs, assign the default value $T_i := K_i$.

This paper extends a recent article [17]. To explain [17], introduce a "level of success" y. In the processes $(K_i, T_i(y), \mathbb{I}(\mathcal{E}_i(y)))$ that result, the successes and the success times (but not the renewal times) depend on the level of success. For consistency with [17], let $T_0 := T_0(y)$, $\mathcal{E}_y^{(0)} := \mathcal{E}_0(y)$, and $\mathcal{E}_-^{(0)} := \mathcal{E}_0^c(y)$ (the complement of $\mathcal{E}_y^{(0)}$). In addition, define the generic cycle representatives $K := K_1 - K_0$, $T := T_1(y) - K_0$, $\mathcal{E}_y := \mathcal{E}_1(y)$, and $\mathcal{E}_- := \mathcal{E}_1^c(y)$. (The dependency of T_0, T, $\mathcal{E}_-^{(0)}$, and $\mathcal{E}_-$ on the parameter y is suppressed.) As a prerequisite to Poisson limit theorems, assume $\lim_{y\to\infty}\mathbb{P}(\mathcal{E}_y) = 0$, where $y \to \infty$ through its permitted values (e.g., real or integer).

Let $\#S$ denote the cardinality of a set S. We shall study the number of successes $N(k) := \#\{i \geqslant 0\colon\ \mathcal{E}_y^{(i)}$ occurs and $T_i \leqslant k\}$ that occur by time k. (For consistency with [17], k represents continuous time.) The number $\tilde{N}(k) := \#\{i \geqslant 1\colon\ \mathcal{E}_y^{(i)}$ occurs and $T_i \leqslant k + K_0\}$ is also useful, because $N(k)$ and $\tilde{N}(k)$ have identical distributions, if the 0th renewal is probabilistically the same as the other renewals.

If k tends to infinity as $\mathbb{P}(\mathcal{E}_y)$ tends to 0, so that $\lambda_0 := \lim_{y\to\infty}(k/\mathbb{E}K)\mathbb{P}(\mathcal{E}_y)$ is a finite real number, then $\lim_{y\to\infty}\mathbb{E}N(k) = \lambda_0$. The renewal set-up leads naturally to Poisson convergence theorems: $\lim_{y\to\infty}\mathbb{P}\{N(k) = j\} = \exp(-\lambda_0)\lambda_0^j/j!$. These theorems also entail exponential limit theorems [12] for the dual variable $T_y^* := \min\{T_i\colon\ \mathcal{E}_y^{(i)}\text{occurs}\}$, because of the equality of events $[T_y^* > k] = [N(k) = 0]$.

Let $\mathbb{E}(K; \mathcal{E}_-) := \mathbb{E}(K|\mathcal{E}_-)\mathbb{P}(\mathcal{E}_-)$, etc. The defective renewal equation

$$\mathbb{P}\bigl\{N(k) = 0\bigr\} = \mathbb{P}\bigl(K_0 > k;\, \mathcal{E}_-^{(0)}\bigr) + \mathbb{P}\bigl(T_0 > k;\, \mathcal{E}_y^{(0)}\bigr) + \int\limits_{[0,k]} \mathbb{P}\bigl\{\tilde{N}(k - x) = 0\bigr\}\mathbb{P}\bigl(K_0 = dx;\, \mathcal{E}_-^{(0)}\bigr) \tag{1.1}$$

is the actual starting point of later analysis. Notice that Eq. (1.1) continues to hold even if cycle independence fails, as long as the process regenerates at K_i $(i = 1, 2, 3, \ldots)$. Thus, the theorems below also hold for regenerative processes with a rare event [7, p. 129].

In bioinformatics, computer simulations of a narrow class of renewal-success processes suggested that a "finite-size correction" (FSC) could sharpen Poisson convergence. Altschul and Gish [3] therefore proposed the FSC as a practical tool in sequence comparison [2]. Presently, computers sharpen sequence comparison statistics with the FSC about once a second [1,4,5]. In fact, the abstract of a recent article stated that any major improvement in sequence comparison statistics probably requires a better understanding of the FSC [14].

In response, [17] showed that the FSC is a phenomenon occurring generally in discrete-time renewal-success processes. The FSC has been applied heuristically in bioinformatics to continuous-time processes, however [10]. Accordingly, this paper extends the underlying theory in [17] (and encounters substantial new analytical difficulties).

Only one of the two assumptions in [17] is used here. The exponential tail condition states that $\mathbb{P}(K \geqslant k) \leqslant Ce^{-\tilde{r}k}$ and $\mathbb{P}(K_0 \geqslant k) \leqslant C_0 e^{-\tilde{r}k}$ for some C, C_0, and $\tilde{r} > 0$. The extra assumption in [17] can slightly strengthen the error bounds in our theorems, but the stronger bounds have no practical importance [15].

Theorems 1.1 and 1.2 below generalize the corresponding theorems in [17]. Throughout this paper, asymptotic statements refer to limit as $y \to \infty$ (implicitly, through permitted values), unless otherwise indicated. We also adopt the Landau O- and o-notation [9]. The constants implicit in the O-notation might vary from equation to equation. Although they might depend on K or on r (defined in Theorem 1.1), they are independent of y or other variables. In contrast, functions in the o-notation are independent of K and r, and they depend only on the indicated arguments.

Theorem 1.1. *Assume the exponential tail condition, and let $z := \mathbb{P}(\mathcal{E}_y)/\mathbb{E}(K; \mathcal{E}_-)$. For a single renewal process operating up to time k, there exists some $r > 0$, such that the probability of having no success is*

$$\mathbb{P}\{N(k) = 0\} = \hat{p}(k)\exp\{O(kz^2) + O(z)\} + O(z) + O(e^{-rk}) \tag{1.2}$$

as $y \to \infty$, where

$$\hat{p}(k) = \mathbb{P}\big(\mathcal{E}_-^{(0)}\big)\exp\big[-z\{k - \mathbb{E}(T|\mathcal{E}_y)\}\big]. \tag{1.3}$$

After writing $\mathbb{E}(K; \mathcal{E}_-) = \mathbb{E}K - \mathbb{E}(K; \mathcal{E}_y)$, routine algebra and the Cauchy–Schwartz inequality $O\{\mathbb{E}(K; \mathcal{E}_y)\}^2 \leqslant O\{\mathbb{E}(K^2; \mathcal{E}_y)\mathbb{P}(\mathcal{E}_y)\} = o\{\mathbb{P}(\mathcal{E}_y)\}$ give the following corollary.

Corollary 1.1. *If $\lambda_0 := \lim_{y\to\infty}(k/\mathbb{E}K)\mathbb{P}(\mathcal{E}_y)$ exists as a finite non-zero real number, then*

$$\mathbb{P}\{N(k) = 0\} = \tilde{p}(k) + O\{\mathbb{P}(\mathcal{E}_y)\} + O(e^{-rk}) \tag{1.4}$$

as $y \to \infty$, where

$$\tilde{p}(k) = \mathbb{P}\big(\mathcal{E}_-^{(0)}\big)\exp\left[-\frac{\mathbb{P}(\mathcal{E}_y)}{\mathbb{E}K}\{k - \mathbb{E}(T|\mathcal{E}_y) + \lambda_0\mathbb{E}(K|\mathcal{E}_y)\}\right]. \tag{1.5}$$

In genomics applications, $\lambda_0 := \lim_{y\to\infty}(k/\mathbb{E}K)\mathbb{P}(\mathcal{E}_y)$ often exists as a small positive constant, with the final term in Eq. (1.5) negligible. Even if the limit λ_0 does not exist, however, Eq. (1.2) makes sense, so the duplication of $O(z)$ there is not redundant.

In the rest of the paper, r retains its value from Theorem 1.1. Consider now A independent, identically distributed copies of the process described above. The copies operate in parallel for durations $0 < k_1 \leqslant k_2 \leqslant k_3 \leqslant \cdots \leqslant k_A$, with $V := \sum_{j=1}^{A} k_j$ (A for "area"; V for "volume"). Define the total number of successes $N := \sum_{j=1}^{A} N_j$, where N_j is the number of successes that the jth process produces in its duration k_j.

Theorem 1.2 below assumes $\mathbb{P}(\mathcal{E}_y^{(0)}) = O\{\mathbb{P}(\mathcal{E}_y)\}$ and $\mathbb{E}(K_0|\mathcal{E}_y^{(0)}) = O\{\mathbb{E}(K^2|\mathcal{E}_y)\}$. Define $\check{k} := \lceil (2/r)\ln V \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x ("ln" is the natural logarithm). Let $\tilde{A} := \#\{j\colon k_j \leqslant \check{k}\}$, the number of processes whose duration does not exceed $\check{k}$, with $\check{A} := \max\{1, \tilde{A}\}$. Moreover, let $Y_{\hat{\lambda}}$ denote a Poisson variable of mean

$$\hat{\lambda} = \frac{\mathbb{P}(\mathcal{E}_y)}{\mathbb{E}K}\bigl\{V - A\mathbb{E}(T|\mathcal{E}_y)\bigr\}. \tag{1.6}$$

Recall that the total variational distance between two real random variables X and Y is $d_{TV}(X, Y) := \sup_A |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|$, the supremum being over real Borel sets A.

Theorem 1.2. *Assume the exponential tail condition, and consider A renewal processes operating in parallel, as described above. Assume $\hat{\lambda}_0 := \lim_{y\to\infty}(V/\mathbb{E}K)\mathbb{P}(\mathcal{E}_y)$ exists as a finite non-zero real number. As $y \to \infty$,*

$$d_{TV}(N, Y_{\hat{\lambda}}) = O(\check{A}V^{-1}\log V) + O\left(V^{-2}\sum_{j=\check{A}+1}^{A} k_j^2\right) + O(AV^{-1}). \tag{1.7}$$

In genomics, the FSC has been applied to p-values that detect promoter motifs in genomic DNA [10]. In conjunction with other unpublished results [18], Theorem 1.2 justifies the application of the FSC to those p-values. Potentially, however, Theorems 1.1 and 1.2 have many other applications. The applications include queuing theory, insurance risk, and traffic flow [17]. Some exponential limit theorems about Markov chains (e.g., [8, Theorem 2], [12, Theorem 8.2B]) are also sharpened by easy corollaries of Theorem 1.1. In addition, systems reliability theory often examines regenerative processes with a rare event. The reliability applications often require tight error bounds [11, p. 152], [12, p. 130], like the ones in Theorems 1.1 and 1.2.

Let $t =: u + iv$ in the rest of this paper. As a prelude to the proofs of Theorems 1.1 and 1.2, Section 2 locates the zeros of $\mathbb{E}e^{tK} - 1 = 0$ near the imaginary axis. Section 3 then proves Eq. (1.2) in Theorem 1.1; Section 4, Eq. (1.7) in Theorem 1.2.

2. The location of the zeros

This section locates the roots ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$ near the imaginary axis. It is structured somewhat formally, with "lemmas" and "propositions." The next two sections require the lemmas, whereas the propositions are merely intermediate results. For easy reference, the lemmas are stated first, their proofs deferred until enough propositions have accumulated. The lemmas contain three quantities $\varepsilon > 0$, $r > 0$, and y. The proofs implicitly determine ε and r, and a lower bound for y, so that the lemmas hold.

In the complex t-plane, let $\overline{S(u_1, u_2)} := \{t\colon\ u_1 \leqslant u \leqslant u_2\}$ be the closed strip between $u = u_1$ and $u = u_2$, and let $\overline{C_r(t_0)} := \{t\colon\ |t - t_0| \leqslant r\}$ be the closed ball of radius r around t_0. Lemmas 2.1 and 2.3 are motivated by the properties of imaginary roots of $\mathbb{E}e^{\zeta K} - 1 = 0$ when K is arithmetic.

Lemma 2.1. *All roots ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$ in $\overline{S(0, 4r)}$ satisfy $|\mathbb{E}(Ke^{\zeta K})| \geqslant \frac{1}{4}\mathbb{E}K$.*

Lemma 2.2. *If $\hat{\zeta} \in \overline{S(0, 3r)}$ satisfies $|\mathbb{E}e^{\hat{\zeta} K} - 1| \leqslant \varepsilon$, then $\overline{C_{r/2}(\hat{\zeta})}$ contains a single root ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$.*

Remark. (1) The "hat" notation is a reminder that $\hat{\zeta}$ is an approximate root of $\mathbb{E}e^{\zeta K} - 1 = 0$.

(2) Lemma 2.2 is used in its converse form: if $t \in \overline{S(0, 3r)}$ and $\overline{C_{r/2}(t)}$ does *not* contain any root ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$, then $|\mathbb{E}e^{tK} - 1| > \varepsilon$.

Lemma 2.3. *Let $\zeta_0 := 0$. All roots ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$ in $\overline{S(0, 4r)}$ can be enumerated as $\ldots, \zeta_{-1}, \zeta_0, \zeta_1, \ldots,$ where $\operatorname{Im}\zeta_m + 2r < \operatorname{Im}\zeta_{m+1}$. (The sequence may terminate on either side. Indeed, it may consist of ζ_0 alone.) Thus, $|\operatorname{Im}\zeta_{\pm m}| > 2r|m|$ for all permissible $m \neq 0$, and in particular, $\sum_{m\neq 0} |\zeta_m^{-2}| < \infty$.*

The final three lemmas are analogs of the first three. They indicate that if y is large enough, the roots τ of $\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1 = 0$ behave almost like the roots ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$.

Lemma 2.4. *All roots τ of $\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1 = 0$ in $\overline{S(0, 3r)}$ satisfy $|\mathbb{E}(Ke^{\tau K}; \mathcal{E}_-)| \geqslant \frac{1}{8}\mathbb{E}K$.*

Lemma 2.5. *If $\hat{\tau} \in \overline{S(0, 3r)}$ satisfies $|\mathbb{E}(e^{\hat{\tau} K}; \mathcal{E}_-) - 1| \leqslant \frac{1}{2}\varepsilon$, then $\overline{C_{r/2}(\hat{\tau})}$ contains a single root ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$.*

Lemma 2.6. *All roots τ of $\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1 = 0$ in $\overline{S(0, 3r)}$ can be enumerated $\ldots, \tau_{-1}, \tau_0, \tau_1, \ldots,$ where $\operatorname{Im}\tau_0 = 0$ and $\operatorname{Im}\tau_m + r < \operatorname{Im}\tau_{m+1}$. (Again, the sequence may terminate on either side or consist of a single root τ_0 alone.) Thus, $|\operatorname{Im}\tau_{\pm m}| > r|m|$ for all permissible $m \neq 0$, and in particular, $\sum_{m\neq 0} |\tau_m^{-2}| < \infty$.*

Proofs are based on the Taylor expansion with Lagrange remainder [20, p. 45] and Rouché's theorem [16, p. 218]. Section 3 and 4 should probably be read first.

The proofs expand and bound generating functions like $\mathbb{E}e^{tK}$, $\mathbb{E}(e^{tK}; \mathcal{E}_y)$, etc., in some narrow strip $\overline{S(0, 4r)}$. Set $\rho < \tilde{r}$. If $9r \leqslant \rho$ and $t \in \overline{S(0, 4r)}$, the generating functions have no poles in the ball $\overline{C_{5r}(t)}$, so Taylor expansions of the generating functions about t have sufficient latitude to converge within $\overline{C_{5r}(t)}$. Moreover, if $\operatorname{Re} t \leqslant \rho$, bounds can be derived by majorizing the corresponding integrands, e.g., $\mathbb{E}(Ke^{tK}) \leqslant \mathbb{E}(Ke^{\rho K})$, $\mathbb{E}(K^2 e^{tK}; \mathcal{E}_y) \leqslant \mathbb{E}(K^2 e^{\rho K}; \mathcal{E}_y)$, etc.

Proposition 2.1. *For some $\varepsilon > 0$ small enough, if $v \in \mathbb{R}$ satisfies $|\mathbb{E}e^{ivK} - 1| \leqslant 2\varepsilon$, then $|\mathbb{E}(Ke^{ivK})| \geqslant \frac{1}{2}\mathbb{E}K$.*

Proof. For any $\delta > 0$, define the event $A_\delta := [\delta \leqslant |1 - e^{ivK}|]$ and its complement A_δ^c. The Chebyshev and Cauchy–Schwarz inequalities give

$$\begin{aligned}\left\{\delta \mathbb{P}(A_\delta)\right\}^2 &\leqslant \left\{\mathbb{E}|1 - e^{ivK}|\right\}^2 \leqslant \mathbb{E}\left\{|1 - e^{ivK}|^2\right\} = 2\left\{1 - \mathbb{E}\cos(vK)\right\} \\ &\leqslant 2|1 - \mathbb{E}e^{ivK}| \leqslant 4\varepsilon. \end{aligned} \tag{2.1}$$

Set $\delta := \varepsilon^{1/4}$ to yield $\mathbb{P}(A_\delta) \leqslant 2\delta$. Thus,

$$\begin{aligned}\left|\mathbb{E}K - \mathbb{E}(Ke^{ivK})\right| &\leqslant \mathbb{E}\left\{|1 - e^{ivK}|K\right\} \leqslant 2\mathbb{E}(K; A_\delta) + \delta\mathbb{E}\left(K; A_\delta^c\right) \\ &\leqslant 2\mathbb{E}(K; A_\delta) + \delta\mathbb{E}K. \end{aligned} \tag{2.2}$$

Because $\mathbb{P}(A_\delta) \leqslant 2\delta$, dominated convergence yields $\lim_{\delta\downarrow 0}\{2\mathbb{E}(K; A_\delta) + \delta\mathbb{E}K\} = 0$. We can therefore select $\delta := \varepsilon^{1/4}$ small enough so that the right side of Eq. (2.2) is less than $\frac{1}{2}\mathbb{E}K$. With this value of ε, $|\mathbb{E}(Ke^{ivK})| \geqslant \mathbb{E}K - |\mathbb{E}K - \mathbb{E}(Ke^{ivK})| \geqslant \frac{1}{2}\mathbb{E}K$. $\square$

Proposition 2.2. *Let $\varepsilon > 0$ be as in Proposition* 2.1. *For some $r > 0$ small enough, if $|\mathbb{E}e^{\hat{\zeta}K} - 1| \leqslant \varepsilon$ for $\hat{\zeta} \in S\overline{(0, 4r)}$, then $|\mathbb{E}(Ke^{\hat{\zeta}K})| \geqslant \frac{1}{4}\mathbb{E}K$.*

Proof. Begin with $r = \frac{1}{9}\rho$, where ρ is defined just above Proposition 2.1, and let $v := \operatorname{Im}\hat{\zeta}$. The Taylor expansion within $\overline{C_{4r}(\hat{\zeta})}$ yields $|\mathbb{E}e^{\hat{\zeta}K} - \mathbb{E}e^{ivK}| \leqslant 4r\mathbb{E}(Ke^{\rho K})$. Reduce $r > 0$ if necessary, so $4r\mathbb{E}(Ke^{\rho K}) \leqslant \varepsilon$. Proposition 2.1 with the inequality $|\mathbb{E}e^{ivK} - 1| \leqslant |\mathbb{E}e^{\hat{\zeta}K} - \mathbb{E}e^{ivK}| + |\mathbb{E}e^{\hat{\zeta}K} - 1| \leqslant 2\varepsilon$ then yield $|\mathbb{E}(Ke^{ivK})| \geqslant \frac{1}{2}\mathbb{E}K$.

Now, consider the Taylor expansion $|\mathbb{E}(Ke^{\hat{\zeta}K}) - \mathbb{E}(Ke^{ivK})| \leqslant 4r\mathbb{E}(K^2e^{\rho K})$. Reduce $r > 0$ if necessary, so that $4r\mathbb{E}(K^2e^{\rho K}) \leqslant \frac{1}{4}\mathbb{E}K$. The inequality $|\mathbb{E}(Ke^{\hat{\zeta}K})| \geqslant |\mathbb{E}(Ke^{ivK})| - |\mathbb{E}(Ke^{\hat{\zeta}K}) - \mathbb{E}(Ke^{ivK})|$ then yields $|\mathbb{E}(Ke^{\hat{\zeta}K})| \geqslant \frac{1}{4}\mathbb{E}K$. $\square$

Proof of Lemma 2.1. Lemma 2.1 follows from Proposition 2.2 with $|\mathbb{E}e^{\zeta K} - 1| = 0 \leqslant \varepsilon$. $\square$

Proof of Lemma 2.2. A Taylor expansion of $\mathbb{E}e^{tK} - 1$ within $\overline{C_{r/2}(\hat{\zeta})}$ yields

$$\left|(\mathbb{E}e^{tK} - 1) - \left\{(\mathbb{E}e^{\hat{\zeta}K} - 1) - (t - \hat{\zeta})\mathbb{E}(Ke^{\hat{\zeta}K})\right\}\right| \leqslant \frac{1}{2}\left(\frac{1}{2}r\right)^2 \mathbb{E}(K^2e^{\rho K}). \tag{2.3}$$

On the other hand, for $|t - \hat{\zeta}| = \frac{1}{2}r$, Proposition 2.2 gives

$$\begin{aligned}\left|(\mathbb{E}e^{\hat{\zeta}K} - 1) - (t - \hat{\zeta})\mathbb{E}(Ke^{\hat{\zeta}K})\right| &\geqslant \left|(t - \hat{\zeta})\mathbb{E}(Ke^{\hat{\zeta}K})\right| - |\mathbb{E}e^{\hat{\zeta}K} - 1| \\ &\geqslant \left(\frac{1}{2}r\right)\left(\frac{1}{4}\mathbb{E}K\right) - \varepsilon. \end{aligned} \tag{2.4}$$

All of the proofs to this point continue to hold through any future reductions in r, if we maintain the relationship $\frac{1}{2}\left(\frac{1}{2}r\right)^2\mathbb{E}(K^2e^{\rho K}) < \left(\frac{1}{2}r\right)\left(\frac{1}{4}\mathbb{E}K\right) - \varepsilon$ between Eqs. (2.3) and (2.4). Thus, the remaining proofs implicitly determine ε from r with the equation

$$\varepsilon = \left(\frac{1}{2}r\right)\left(\frac{1}{4}\mathbb{E}K\right) - \left(\frac{1}{2}r\right)^2 \mathbb{E}(K^2e^{\rho K}). \tag{2.5}$$

Two initial reductions ensure that previous proofs continue to hold. First, decrease $r > 0$ if necessary, so $r \leqslant \left(\frac{1}{4}\mathbb{E}K\right)/\mathbb{E}(K^2 e^{\rho K})$. Thus, further reductions in r also decrease ε, because $d\varepsilon/dr = \left(\frac{1}{2}\right)\left(\frac{1}{4}\mathbb{E}K\right) - \left(\frac{1}{2}r\right)\mathbb{E}(K^2 e^{\rho K}) \geqslant 0$. Second, decrease r further if necessary, so $\varepsilon > 0$ is at least as small as it was before Eq. (2.5).

Because of Eqs. (2.3)–(2.5), Rouché's theorem shows that the equations $\mathbb{E}e^{tK} - 1 = 0$ and $(\mathbb{E}e^{\hat{\zeta}K} - 1) - (t - \hat{\zeta})\mathbb{E}(Ke^{\hat{\zeta}K}) = 0$ have the same number of roots in $\overline{C_{r/2}(\hat{\zeta})}$. The root t of the second equation satisfies $|t - \hat{\zeta}| \leqslant \varepsilon/|\mathbb{E}(Ke^{\hat{\zeta}K})| \leqslant \varepsilon/\left(\frac{1}{4}\mathbb{E}K\right) < \frac{1}{2}r$, so $\overline{C_{r/2}(\hat{\zeta})}$ contains a single root of each equation, in particular a single, simple root ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$. $\square$

In future, propositions do not mention the value of r explicitly. Implicitly, r starts with its prior value and decreases as required by the proofs.

Proposition 2.3. *If $\zeta, \zeta' \in S\overline{(0, 4r)}$ are distinct roots of $\mathbb{E}e^{\zeta K} - 1 = 0$, then $|\operatorname{Im}(\zeta - \zeta')| > 2r$.*

Proof. Let $\zeta \in S\overline{(0, 4r)}$ satisfy $\mathbb{E}e^{\zeta K} - 1 = 0$. Because $|\mathbb{E}e^{\zeta K} - 1| = 0 \leqslant \varepsilon$, Lemma 2.1 shows that $|\mathbb{E}(Ke^{\zeta K})| \geqslant \frac{1}{4}\mathbb{E}K$. A Taylor expansion of $\mathbb{E}e^{tK}$ within $\overline{C_{5r}(\zeta)}$ yields $|\mathbb{E}e^{tK} - 1 - (t - \zeta)\mathbb{E}(Ke^{\zeta K})| \leqslant |t - \zeta|^2 \mathbb{E}(K^2 e^{\rho K})$. Thus,

$$\begin{aligned}
|\mathbb{E}e^{tK} - 1| &\geqslant \left|(t - \zeta)\mathbb{E}(Ke^{\zeta K})\right| - \left|\mathbb{E}e^{tK} - 1 - (t - \zeta)\mathbb{E}(Ke^{\zeta K})\right| \\
&\geqslant |t - \zeta| \left\{\left|\mathbb{E}(Ke^{\zeta K})\right| - |t - \zeta|\mathbb{E}(K^2 e^{\rho K})\right\} \\
&\geqslant |t - \zeta| \left\{\tfrac{1}{4}\mathbb{E}K - |t - \zeta|\mathbb{E}(K^2 e^{\rho K})\right\}. \qquad (2.6)
\end{aligned}$$

If $0 < |t - \zeta| < 4\mathbb{E}(K^2 e^{\rho K})/\mathbb{E}K$, then $\mathbb{E}e^{tK} \neq 1$. Continue the reduction strategy: reduce $r > 0$ if necessary, so that $5r \leqslant \mathbb{E}(K^2 e^{\rho K})/\left(\frac{1}{4}\mathbb{E}K\right)$. From Eq. (2.6), each pair of roots ζ and ζ' of $\mathbb{E}e^{\zeta K} - 1 = 0$ within $S\overline{(0, 4r)}$ then must satisfy $|\zeta' - \zeta| \geqslant 5r$. Because $|\operatorname{Re}(\zeta' - \zeta)| \leqslant 4r$, we find $|\operatorname{Im}(\zeta' - \zeta)|^2 \geqslant (5r)^2 - (4r)^2 > 4r^2$. $\square$

Proof of Lemma 2.3. Start at $\zeta_0 := 0$ and proceed in a positive imaginary direction, enumerating in order any roots ζ_m of $\mathbb{E}e^{\zeta K} - 1 = 0$ encountered in the strip $S\overline{(0, 4r)}$ ($m = 1, 2, 3, \ldots$). Proposition 2.3 ensures that the enumeration is feasible, because the roots in question have imaginary parts differing by at least $2r$. Because the conjugates $\bar{\zeta}$ of the roots satisfy $\mathbb{E}e^{\bar{\zeta}K} - 1 = \overline{\mathbb{E}e^{\zeta K} - 1} = 0$, define $\zeta_{-m} := \bar{\zeta}_m$ to complete the enumeration in the strip $S\overline{(0, 4r)}$. Proposition 2.3 shows that $\operatorname{Im}\zeta_m + 2r < \operatorname{Im}\zeta_{m+1}$, so $|\operatorname{Im}\zeta_{\pm m}| > 2r|m|$ for $m \neq 0$ by induction in the positive and negative imaginary directions. In addition, $\sum_{m \neq 0} |\zeta_m^{-2}| < (2r)^{-2} 2 \sum_{m=1}^{\infty} m^{-2} < \infty$. $\square$

The values of ε and r are static in the rest of this section. We now increase y progressively, if necessary. Like our conventions concerning r above, our proofs below do not mention any initial lower bound for y explicitly. Implicitly, the lower bound for y starts with its prior value and increases as the proofs require.

Proof of Lemma 2.4. $\limsup_{y\to\infty} |\mathbb{E}(e^{\tau K}; \mathcal{E}_y)| \leqslant \limsup_{y\to\infty} \mathbb{E}(e^{\rho K}; \mathcal{E}_y) = 0$ by dominated convergence. For y large enough, $|\mathbb{E}(e^{\rho K}; \mathcal{E}_y)| \leqslant \varepsilon$. Thus,

$$|\mathbb{E}e^{\tau K} - 1| \leqslant \left|\mathbb{E}(e^{\tau K}; \mathcal{E}_y)\right| + \left|\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1\right| \leqslant \varepsilon,$$

so $|\mathbb{E}(Ke^{\tau K})| \geqslant \frac{1}{4}\mathbb{E}K$ by Proposition 2.2. Again,

$$\limsup_{y\to\infty}\left|\mathbb{E}(Ke^{\tau K}; \mathcal{E}_y)\right| \leqslant \limsup_{y\to\infty} \mathbb{E}(Ke^{\rho K}; \mathcal{E}_y) = 0$$

by dominated convergence. Increase y again if necessary, so that $|\mathbb{E}(Ke^{\rho K}; \mathcal{E}_y)| \leqslant \frac{1}{8}\mathbb{E}K$. Lemma 2.4 follows, because of the triangle inequality $|\mathbb{E}(Ke^{\tau K}; \mathcal{E}_-)| \geqslant |\mathbb{E}(Ke^{\tau K})| - |\mathbb{E}(Ke^{\tau K}; \mathcal{E}_y)| \geqslant \frac{1}{8}\mathbb{E}K$. $\square$

Proof of Lemma 2.5. Increase y if necessary, so that dominated convergence makes $|\mathbb{E}(e^{\rho K}; \mathcal{E}_y)| \leqslant \frac{1}{2}\varepsilon$. Thus, $|\mathbb{E}e^{\hat{\tau} K} - 1| \leqslant |\mathbb{E}(e^{\hat{\tau} K}; \mathcal{E}_y)| + |\mathbb{E}(e^{\hat{\tau} K}; \mathcal{E}_-) - 1| \leqslant \varepsilon$. By Lemma 2.2, $\overline{C_{r/2}(\hat{\tau})}$ contains a single root ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$. $\square$

Proposition 2.4. *If $\zeta \in S\overline{(0, 4r)}$ satisfies $\mathbb{E}e^{\zeta K} - 1 = 0$, then $\overline{C_{r/2}(\zeta)}$ contains a single, simple root τ of $\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1 = 0$.*

Proof. Note

$$\left|\left\{\mathbb{E}(e^{tK}; \mathcal{E}_-) - 1\right\} - (\mathbb{E}e^{tK} - 1)\right| \leqslant \mathbb{E}(e^{\rho K}; \mathcal{E}_y). \tag{2.7}$$

Because $\mathbb{E}e^{\zeta K} - 1 = 0$, a Taylor expansion of $\mathbb{E}e^{tK} - 1$ within $\overline{C_{r/2}(\zeta)}$ yields $|\mathbb{E}e^{tK} - 1 + (t-\zeta)\mathbb{E}(Ke^{\zeta K})| \leqslant \frac{1}{2}\left(\frac{1}{2}r\right)^2\mathbb{E}(K^2e^{\rho K})$. For $|t-\zeta| = \frac{1}{2}r$,

$$\begin{aligned}|\mathbb{E}e^{tK} - 1| &\geqslant \left(\frac{1}{2}r\right)\left|\mathbb{E}(Ke^{\zeta K})\right| - \frac{1}{2}\left(\frac{1}{2}r\right)^2\mathbb{E}(K^2e^{\rho K})\\ &\geqslant \left(\frac{1}{2}r\right)\left(\frac{1}{4}\mathbb{E}K\right) - \frac{1}{2}\left(\frac{1}{2}r\right)^2\mathbb{E}(K^2e^{\rho K}),\end{aligned} \tag{2.8}$$

where the triangle inequality applied to the Taylor expansion justifies the first inequality. The second inequality follows from Proposition 2.2. Increase y if necessary, so that $\mathbb{E}(e^{\rho K}; \mathcal{E}_y) < \left(\frac{1}{2}r\right)\left(\frac{1}{4}\mathbb{E}K\right) - \frac{1}{2}\left(\frac{1}{2}r\right)^2\mathbb{E}(K^2e^{\rho K})$. Rouché's theorem applied to Eqs. (2.7) and (2.8) then shows that the equations $\mathbb{E}e^{tK} - 1 = 0$ and $\mathbb{E}(e^{tK}; \mathcal{E}_-) - 1 = 0$ have the same number of roots in $\overline{C_{r/2}(\zeta)}$. $\square$

Proposition 2.5. *If $\tau \in S\overline{(0, 3r)}$ satisfies $\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1 = 0$, then $\overline{C_{r/2}(\tau)}$ contains a single root ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$.*

Proof. Because $|\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1| = 0 \leqslant \frac{1}{2}\varepsilon$, Proposition 2.5 follows immediately from Lemma 2.5. $\square$

Proof of Lemma 2.6. For large enough y, the equation $\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1 = 0$ has a positive, real root, because $\mathbb{P}(\mathcal{E}_-) < 1 < \mathbb{E}e^{rK} = \lim_{y\to\infty}\mathbb{E}(e^{rK}; \mathcal{E}_-)$ by dominated convergence.

We call the root $\tau_0 := \tau_0(y) > 0$. The root τ_0 is the only positive real root, because $f(\tau) := \mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1$ is strictly increasing: $f'(\tau) := \mathbb{E}(Ke^{\tau K}; \mathcal{E}_-) \geqslant \mathbb{E}(K; \mathcal{E}_-) > 0$ for large enough y by dominated convergence.

Now, note that $S\overline{\big(0, 3r + \frac{1}{2}r\big)} \subset S\overline{(0, 4r)}$. Proposition 2.5 shows that for each root $\tau \in S\overline{(0, 3r)}$, there is a ζ satisfying $\mathbb{E}e^{\zeta K} - 1 = 0$ within $\overline{C_{r/2}(\tau)}$. Moreover, τ is the only root whose ball $\overline{C_{r/2}(\tau)}$ contains ζ, because Proposition 2.4 shows that τ is the only root within $\overline{C_{r/2}(\zeta)}$. Thus, the roots τ are in one-to-one correspondence with a subsequence $\ldots, \zeta_{j(-1)}, \zeta_{j(0)}, \zeta_{j(1)}, \ldots$ of $\ldots, \zeta_{-1}, \zeta_0, \zeta_1, \ldots$. (Again, the subsequence $\ldots, \zeta_{j(-1)}, \zeta_{j(0)}, \zeta_{j(1)}, \ldots$ may terminate on either side or consist of $\zeta_0 := 0$ alone.) Renumber the subsequence $\ldots, \zeta_{j(-1)}, \zeta_{j(0)}, \zeta_{j(1)}, \ldots$ if necessary, so that $\zeta_{j(0)}$ corresponds to τ_0, and then number the roots τ, so that τ_m corresponds to $\zeta_{j(m)}$. The inclusion $\tau_m \in \overline{C_{r/2}(\zeta_{j(m)})}$ implies $\operatorname{Im}\tau_m + r \leqslant \operatorname{Im}\zeta_{j(m)} + \frac{3}{2}r < \operatorname{Im}\zeta_{j(m+1)} - \frac{1}{2}r \leqslant \operatorname{Im}\tau_{m+1}$ because of Lemma 2.3, so $\operatorname{Im}\tau_m + r < \operatorname{Im}\tau_{m+1}$. Recall $\operatorname{Im}\tau_0 = 0$. By induction, as in the proof of Lemma 2.3, $|\operatorname{Im}\tau_{\pm m}| > r|m|$. Thus, $\sum_{m \neq 0} |\tau_m^{-2}| < \infty$. $\square$

3. The finite-size correction in a single process

This section derives the finite-size correction for $\mathbb{P}\{N(k) = 0\}$ given in Theorem 1.1 for a single renewal-success process. It uses the lemmas of Section 2 freely, assuming implicitly appropriate values for ε and r, and an appropriate lower bound for y.

Recall the Introduction's definition $N(k) := \#\{i \geqslant 0\colon\ \mathcal{E}_y^{(i)}$ occurs and $T_i \leqslant k\}$. Define the moment generating function

$$\varphi(s, t) := \int_0^\infty e^{tk} \left[\sum_{j=0}^\infty \mathbb{P}\big\{N(k) = j\big\} s^j \right] dk. \tag{3.1}$$

The moment generating function converges in a region containing the set $\{(s, t) \in \mathbb{C}^2\colon |s| \leqslant 1,\ \operatorname{Re} t < 0\}$. Because Eq. (3.1) also holds for 0-delayed renewals, it holds with $\tilde{\varphi}(s, t)$ and $\tilde{N}(k)$ replacing $\varphi(s, t)$ and $N(k)$.

The moment generating functions and the integral over e^{kt} in Eq. (3.1) are analogous to the probability generating functions and sum over t^k in [17]. Similarly, the Introduction in this paper substituted $O(e^{-rk})$ for the analogous $O(r^{-k})$ in [17]. In the following, non-arithmetic variables consistently replace their arithmetic counterparts in [17], leading to many cosmetic changes. The following therefore gives its proofs in detail only where they present novel difficulties. Elsewhere, we rely on the proofs in [17].

Define the moment generating function $\mathbb{E}(e^{tK}; \mathcal{E}_y) := \int e^{tk}\mathbb{P}(K = dk; \mathcal{E}_y)$, where the measure $\mathbb{P}(K \leqslant k; \mathcal{E}_y) := \mathbb{P}(K \leqslant k | \mathcal{E}_y)\mathbb{P}(\mathcal{E}_y)$, etc. In [17], arithmetic variables led to the summation identity

$$\sum_{k=0}^\infty \mathbb{P}(T \leqslant k; \mathcal{E}_y) t^k = (1 - t)^{-1} \mathbb{E}(t^T; \mathcal{E}_y), \tag{3.2}$$

valid when $|t| < 1$. Here, non-arithmetic variables lead to the integration identity

$$\int_0^\infty e^{tk}\mathbb{P}(T \leqslant k; \mathcal{E}_y)\,dk = \int_0^\infty e^{tk}\mathbb{E}\{\mathbb{I}(T \leqslant k)\mathbb{I}(\mathcal{E}_y)\}\,dk = \mathbb{E}\left\{\mathbb{I}(\mathcal{E}_y)\int_T^\infty e^{tk}\,dk\right\}$$
$$= -t^{-1}\mathbb{E}\{\mathbb{I}(\mathcal{E}_y)e^{tT}\} = -t^{-1}\mathbb{E}(e^{tT}; \mathcal{E}_y), \tag{3.3}$$

valid for $\operatorname{Re} t < 0$. The interchange of integration in the second equality is justified by Fubini's theorem [16, p. 140], because the double integral in the third expression is absolutely convergent for $\operatorname{Re} t < 0$.

The combinatorial arguments in [17] lend themselves unchanged to present purposes, although we now substitute Eq. (3.3) for Eq. (3.2). Equation (1.1) yields

$$\begin{aligned}\varphi(s,t) &= \big[\mathbb{E}\big(e^{tK_0}; \mathcal{E}_-^{(0)}\big) + s\mathbb{E}\big(e^{tK_0}; \mathcal{E}_y^{(0)}\big)\big]\tilde{\varphi}(s,t) \\ &\quad - t^{-1}\big[\big\{1 - \mathbb{E}\big(e^{tK_0}; \mathcal{E}_-^{(0)}\big) - \mathbb{E}\big(e^{tT_0}; \mathcal{E}_y^{(0)}\big)\big\} \\ &\quad + s\big\{\mathbb{E}\big(e^{tT_0}; \mathcal{E}_y^{(0)}\big) - \mathbb{E}\big(e^{tK_0}; \mathcal{E}_y^{(0)}\big)\big\}\big]. \end{aligned}\tag{3.4}$$

If the renewal in Eq. (3.4) is not delayed, subscripts and superscripts containing 0 can be dropped, e.g., $\mathbb{E}(e^{tK_0}; \mathcal{E}_-^{(0)})$ becomes $\mathbb{E}(e^{tK}; \mathcal{E}_-)$. In that case, however, recall that $\tilde{\varphi}(s,t) = \varphi(s,t)$. The solution of the resulting equation for $\tilde{\varphi}(s,t)$ is

$$\tilde{\varphi}(s,t) = -t^{-1}\left\{1 + \frac{(s-1)\mathbb{E}(e^{tT}; \mathcal{E}_y)}{1 - \mathbb{E}e^{tK} - (s-1)\mathbb{E}(e^{tK}; \mathcal{E}_y)}\right\}, \tag{3.5}$$

which gives a closed solution for $\varphi(s,t)$ upon substitution back into Eq. (3.4).

The combinatorial formula for the generating function of $\mathbb{P}\{N(k) = 0\}$ is therefore

$$\begin{aligned}\varphi(0,t) &= \int_0^\infty e^{tk}\mathbb{P}\{N(k) = 0\}\,dk \\ &= -t^{-1}\bigg[\mathbb{E}\big(e^{tK_0}; \mathcal{E}_-^{(0)}\big)\frac{1 - \mathbb{E}(e^{tK}; \mathcal{E}_-) - \mathbb{E}(e^{tT}; \mathcal{E}_y)}{1 - \mathbb{E}(e^{tK}; \mathcal{E}_-)} \\ &\qquad + \big\{1 - \mathbb{E}\big(e^{tK_0}; \mathcal{E}_-^{(0)}\big) - \mathbb{E}\big(e^{tT_0}; \mathcal{E}_y^{(0)}\big)\big\}\bigg]. \end{aligned}\tag{3.6}$$

Equation (3.6) is valid for $\operatorname{Re} t < 0$, because Fubini's theorem justifies the rearrangement of any convolutions. Equation (3.6) is similar to its counterpart in [17], so its derivation is omitted.

To circumvent some later technical difficulties, we continue our work with

$$\begin{aligned}\varphi_0(t) &:= \int_0^\infty e^{tk}\big[\mathbb{P}\{N(k) = 0\} - \mathbb{P}\big(K_0 > k; \mathcal{E}_-^{(0)}\big) - \mathbb{P}\big(T_0 > k; \mathcal{E}_y^{(0)}\big)\big]\,dk \\ &= -t^{-1}\mathbb{E}\big(e^{tK_0}; \mathcal{E}_-^{(0)}\big)\frac{1 - \mathbb{E}(e^{tK}; \mathcal{E}_-) - \mathbb{E}(e^{tT}; \mathcal{E}_y)}{1 - \mathbb{E}(e^{tK}; \mathcal{E}_-)}, \end{aligned}\tag{3.7}$$

where Eq. (3.6) and equations like Eq. (3.3) yield the second equality. The terms deleted in passing from $\varphi(0,t)$ to $\varphi_0(t)$ contribute a negligible term

$$\mathbb{P}\big(K_0 > k; \mathcal{E}_-^{(0)}\big) + \mathbb{P}\big(T_0 > k; \mathcal{E}_y^{(0)}\big) \leqslant \mathbb{P}(K_0 > k) \leqslant e^{-\tilde{r}k}\mathbb{E}e^{\tilde{r}K_0} = O(e^{-\tilde{r}k}) \tag{3.8}$$

to Eq. (3.6). The first inequality follows from $T_0 \leqslant K_0$; the second, from Chebyshev's.

Let "a.e." abbreviate "almost everywhere under Lebesgue measure on $\mathbb{R}$," and for any $c > 0$, let $V^{\uparrow}(-c, \pm iw)$ indicate the closed line segment directed vertically from $-c - iw$ to $-c + iw$. For sufficiently large y, I claim

$$\mathbb{P}\{N(k) = 0\} - \mathbb{P}(K_0 > k; \mathcal{E}_-^{(0)}) - \mathbb{P}(T_0 > k; \mathcal{E}_y^{(0)}) \\ = \lim_{n\to\infty} (2\pi i)^{-1} \int_{V^{\uparrow}(-c,\pm iw_n)} e^{-kt}\varphi_0(t)\,dt \tag{3.9}$$

for $k \geqslant 0$ a.e. In Eq. (3.9), $\{w_n > 0\}$ is a sequence satisfying $\lim_{n\to\infty} w_n = \infty$.

To justify Eq. (3.9), write $t = -c + iv$ in the integral. Equation (3.7) shows that $\varphi_0(-c + iv)$ is the Fourier transform of e^{-ck} times the left side of Eq. (3.9). Thus, Eq. (3.9) follows from Plancherel's inversion theorem for Fourier transforms in L^2 (e.g., [16, p. 187]), once we show that $|\varphi_0(-c + iv)|^2$ is integrable as a function of v. Because $t = -c + iv$,

$$\begin{aligned} |\varphi_0(-c + iv)| &= |t^{-1}| \frac{|\mathbb{E}(e^{tK_0}; \mathcal{E}_-^{(0)})||1 - \mathbb{E}(e^{tK}; \mathcal{E}_-) - \mathbb{E}(e^{tT}; \mathcal{E}_y)|}{|1 - \mathbb{E}(e^{tK}; \mathcal{E}_-)|} \\ &\leqslant \frac{1}{(c^2 + v^2)^{1/2}} \frac{\mathbb{E}(e^{-cK_0}; \mathcal{E}_-^{(0)})\{1 + \mathbb{E}(e^{-cK}; \mathcal{E}_-) + \mathbb{E}(e^{-cT}; \mathcal{E}_y)\}}{1 - \mathbb{E}(e^{-cK}; \mathcal{E}_-)} \\ &\leqslant \frac{1}{(c^2 + v^2)^{1/2}} \frac{\mathbb{P}(\mathcal{E}_-^{(0)}) \times 2}{1 - \mathbb{P}(\mathcal{E}_-)}. \end{aligned} \tag{3.10}$$

The inequalities follow, because the generating functions in Eq. (3.10) can be majorized: $|\mathbb{E}(e^{tK_0}; \mathcal{E}_-^{(0)})| \leqslant \mathbb{E}(e^{-cK_0}; \mathcal{E}_-^{(0)})$, etc. Because $\int_{-\infty}^{\infty} |\varphi_0(-c + iv)|^2\,dv$ is dominated by a multiple of $\int_{-\infty}^{\infty} (c^2 + v^2)^{-1}\,dv < \infty$, Plancherel's theorem proves Eq. (3.9).

Set $c = 2r$, to permit symmetries to compress later notations. To derive the asymptotics of Eq. (3.9), we apply Cauchy's theorem,

$$(2\pi i)^{-1} \oint_C e^{-kt}\varphi_0(t)\,dt = \sum_{(\tau \text{ in } C)} \operatorname*{Res}_{t=\tau}\{e^{-kt}\varphi_0(t)\}. \tag{3.11}$$

In Eq. (3.11), C is a closed contour oriented counterclockwise, no poles of $e^{-kt}\varphi_0(t)$ lie on C, and the summation takes place over the poles τ of $e^{-kt}\varphi_0(t)$ enclosed by C.

A natural candidate for C in Eq. (3.11) is the closed, counterclockwise oriented, rectangular contour $C = R(\pm 2r, \pm iw_n)$ shown in Fig. 1. Although we want to apply Eq. (3.11) directly to the contour $C = R(\pm 2r, \pm iw_n)$, Eq. (3.11) fails if poles of $e^{-kt}\varphi_0(t)$ lie on $C = R(\pm 2r, \pm iw_n)$. Thus, we restrict the sequence $\{w_n\}$ and then modify the contours $V^{\uparrow}(2r, \pm iw_n)$, so that no poles of $e^{-kt}\varphi_0(t)$ lie on the modification of $R(\pm 2r, \pm iw_n)$.

Because of Lemma 2.3, $\operatorname{Im}\zeta_m + 2r < \operatorname{Im}\zeta_{m+1}$. We can therefore choose a sequence $\{w_n\} \uparrow \infty$, so that $\inf_{m\geqslant 0,\, n>0} |w_n - \operatorname{Im}\zeta_m| = \inf_{m\geqslant 0,\, n>0} |\pm w_n - \operatorname{Im}\zeta_{\pm m}| \geqslant r$. I claim

$$\lim_{n\to\infty} \int_{H^{\leftarrow}(\pm 2r, iw_n)} e^{-kt}\varphi_0(t)\,dt = \lim_{n\to\infty} \int_{H^{\rightarrow}(\pm 2r, -iw_n)} e^{-kt}\varphi_0(t)\,dt = 0. \tag{3.12}$$

(Figure 1 defines various contours, including $H^{\leftarrow}(\pm 2r, iw_n)$ and $H^{\rightarrow}(\pm 2r, -iw_n)$ in Eq. (3.12).) Equation (3.12) follows readily from Eq. (3.7) as follows. For $H^{\leftarrow}(\pm 2r, iw_n)$,

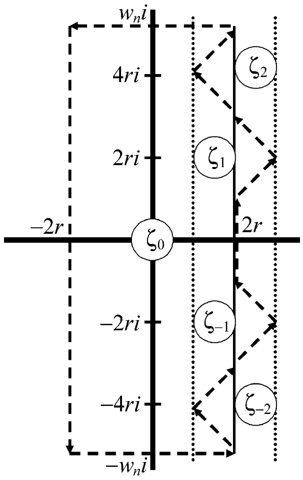

Fig. 1. The oriented rectangular contour $R(\pm 2r, \pm iw_n)$ and the modified contour $\hat{R}(\pm 2r, \pm iw)$. The original rectangular contour $R(\pm 2r, \pm iw_n)$ consists of four edges: (1) $V^{\uparrow}(2r, \pm iw_n)$, the upward solid line segment from $2r - iw_n$ to $2r + iw_n$; (2) $H^{\leftarrow}(\pm 2r, iw_n)$, the leftward dashed line segment from $2r + iw_n$ to $-2r + iw_n$; (3) $V^{\downarrow}(-2r, \pm iw_n)$, the downward dashed line segment from $-2r + iw_n$ to $-2r - iw_n$; and (4) $H^{\rightarrow}(\pm 2r, -iw_n)$, the rightward dashed line segment from $-2r - iw_n$ to $2r - iw_n$. At the centers of the open circles of radius $\frac{1}{2}r$ are the roots ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$. To avoid the roots ζ, the solid line $V^{\uparrow}(2r, \pm iw_n)$ is modified as follows into the dashed line $\hat{V}^{\uparrow}(2r, \pm iw_n)$, which still lies between the dotted lines, within the strip $S\overline{(r, 3r)}$. For any root ζ_m of $\mathbb{E}e^{\zeta K} - 1 = 0$ satisfying $r < \operatorname{Re}\zeta_m < 3r$ and $-w_n < \operatorname{Im}\zeta_m < w_n$, take the line segment in $V^{\uparrow}(2r, \pm iw_n)$ that joins $2r + i(\operatorname{Im}\zeta_m - r)$ and $2r + i(\operatorname{Im}\zeta_m + r)$, and replace it with two line segments. On one hand, if $r < \operatorname{Re}\zeta_m \leqslant 2r$, one replacement segment joins $2r + i(\operatorname{Im}\zeta_m - r)$ and $3r + i\operatorname{Im}\zeta_m$; the other replacement segment, $3r + i\operatorname{Im}\zeta_m$ and $2r + i(\operatorname{Im}\zeta_m + r)$. On the other hand, if $2r < \operatorname{Re}\zeta_m < 3r$, one replacement segment joins $2r + i(\operatorname{Im}\zeta_m - r)$ and $r + i\operatorname{Im}\zeta_m$; the other replacement segment, $r + i\operatorname{Im}\zeta_m$ and $2r + i(\operatorname{Im}\zeta_m + r)$. (Thus, if ζ_m is within distance r of $V^{\uparrow}(2r, \pm iw_n)$ and to the left of it, $V^{\uparrow}(2r, \pm iw_n)$ is diverted to the right, and similarly for the mirrored configuration.) The modified contour $\hat{R}(\pm 2r, \pm iw)$ is the complete dashed contour, consisting of the four edges $\hat{V}^{\uparrow}(2r, \pm iw)$, $H^{\leftarrow}(\pm 2r, iw)$, $V^{\downarrow}(-2r, \pm iw)$, and $H^{\rightarrow}(\pm 2r, -iw)$.

$$\begin{aligned}
&\left| \int_{H^{\leftarrow}(\pm 2r, iw_n)} e^{-kt}\varphi_0(t)\,dt \right| \\
&\quad \leqslant \int_{H^{\leftarrow}(\pm 2r, iw_n)} |e^{-kt}||t^{-1}| \frac{|\mathbb{E}(e^{tK_0}; \mathcal{E}_-^{(0)})||1 - \mathbb{E}(e^{tK}; \mathcal{E}_-) - \mathbb{E}(e^{tT}; \mathcal{E}_y)|}{|1 - \mathbb{E}(e^{tK}; \mathcal{E}_-)|}\,|dt| \\
&\quad \leqslant \left|w_n^{-1}\right| \frac{\mathbb{E}(e^{2rK_0}; \mathcal{E}_-^{(0)})\{1 + \mathbb{E}(e^{2rK}; \mathcal{E}_-) + \mathbb{E}(e^{2rT}; \mathcal{E}_y)\}}{\frac{1}{2}\varepsilon} \int_{H^{\leftarrow}(\pm 2r, iw_n)} |e^{-kt}|\,|dt|.
\end{aligned} \tag{3.13}$$

The pre-factor t^{-1} satisfies $|t^{-1}| \leqslant w_n^{-1}$. As in Eq. (3.10), majorization provides uniform bounds over $H^{\leftarrow}(\pm 2r, iw_n)$ on the absolute values of the numerator. In addition, the distance from any root ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$ to $H^{\leftarrow}(\pm 2r, iw_n)$ or $H^{\rightarrow}(\pm 2r, -iw_n)$ is at least r by construction, so Lemma 2.5 shows that the denominator $|\mathbb{E}(e^{tK}; \mathcal{E}_-) - 1| > \frac{1}{2}\varepsilon$. The final integral in Eq. (3.13) is finite and independent of n. Thus, the common value in

Eq. (3.13) is $O(w_n^{-1})$ and vanishes as $n \to \infty$. The analysis for $H^{\rightarrow}(\pm 2r, -iw_n)$ is similar.

The legend of Fig. 1 shows how to modify the edge $V^{\uparrow}(2r, \pm iw_n)$ of $R(\pm 2r, \pm iw_n)$ to avoid poles τ of $e^{-kt}\varphi_0(t)$. Because $\inf_{m \geqslant 0, n>0} |\pm w_n - \operatorname{Im}\zeta_{\pm m}| \geqslant r$, the directed piecewise linear path $\hat{V}^{\uparrow}(2r, \pm iw_n)$ that results from modifying $V^{\uparrow}(2r, \pm iw_n)$ has the same end-points, namely, $2r \pm iw_n$. Moreover, by construction, the distance between any point t on $\hat{V}^{\uparrow}(2r, \pm iw_n)$ and any root ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$ is greater than $r\sqrt{2}/2$.

Thus, let $C := \hat{R}(\pm 2r, \pm iw)$ in Eq. (3.11) be the closed, directed contour consisting of the four subcontours $\hat{V}^{\uparrow}(2r, \pm iw)$, $H^{\leftarrow}(\pm 2r, iw)$, $V^{\downarrow}(-2r, \pm iw)$, and $H^{\rightarrow}(\pm 2r, -iw)$ in Fig. 1. The summation of the residues in Eq. (3.11) takes place over the poles τ of $e^{-kt}\varphi_0(t)$ enclosed by $\hat{R}(\pm 2r, \pm iw)$. Lemma 2.5 shows that if $\tau = \tau(y) \in \overline{S(0, 3r)}$ satisfies $\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1 = 0$, it is within a distance $\frac{1}{2}r$ of some root ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$. By construction, however, no root ζ can lie within distance $r\sqrt{2}/2 > \frac{1}{2}r$ of $\hat{R}(\pm 2r, \pm iw)$. In view of Eq. (3.12), Cauchy's theorem in Eq. (3.11) yields

$$
\begin{aligned}
&\lim_{n\to\infty} (2\pi i)^{-1} \int\limits_{V^{\uparrow}(-2r, \pm iw_n)} e^{-kt}\varphi_0(t)\,dt \\
&\quad = \lim_{n\to\infty} \left[-\sum_{(\tau)} \operatorname*{Res}_{t=\tau} \bigl\{ e^{-kt}\varphi_0(t) \bigr\} + (2\pi i)^{-1} \int\limits_{\hat{V}^{\uparrow}(2r, \pm iw_n)} e^{-kt}\varphi_0(t)\,dt \right], \qquad (3.14)
\end{aligned}
$$

where the summation is over all roots τ of $\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1 = 0$ enclosed by $\hat{R}(\pm 2r, \pm iw)$. (The residue summation does not include the removable singularity $t = 0$ of $e^{-kt}\varphi_0(t)$.)

The methods in [17] evaluated the residue terms in Eq. (3.14). For any root τ of $\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1 = 0$, the result is

$$
-\operatorname*{Res}_{t=\tau} \bigl\{ e^{-kt}\varphi_0(t) \bigr\} = e^{-\tau k}\tau^{-1} \frac{\mathbb{E}(e^{\tau K_0}; \mathcal{E}_-^{(0)})\mathbb{E}(e^{\tau T}; \mathcal{E}_y)}{\mathbb{E}(K e^{\tau K}; \mathcal{E}_-)}. \qquad (3.15)
$$

The proof of Lemma 2.6 showed that for y large enough, $\mathbb{E}(e^{\tau K}; \mathcal{E}_-) - 1 = 0$ has a unique real root $\tau_0 := \tau_0(y) > 0$. An abbreviated perturbation analysis for τ_0, similar to the analysis in [17], follows. (Detailed analysis of the other roots is unnecessary.)

Recall $z := \mathbb{P}(\mathcal{E}_y)/\mathbb{E}(K; \mathcal{E}_-)$. Because $\mathbb{P}(\mathcal{E}_-) \leqslant \mathbb{P}(\mathcal{E}_-) + \tau_0\mathbb{E}(K; \mathcal{E}_-) \leqslant \mathbb{E}(e^{\tau_0 K}; \mathcal{E}_-) = 1$, we have $\tau_0 = O(z)$. A Taylor expansion of $\mathbb{E}(e^{tK}; \mathcal{E}_-) - 1 = 0$ at $t = \tau_0$ yields

$$
\mathbb{P}(\mathcal{E}_-) + \tau_0\mathbb{E}(K; \mathcal{E}_-) + O\bigl(\tau_0^2\bigr) = 1.
$$

Thus, $\tau_0 = z + O(z^2)$. As in [17], when $\tau_0 = z + O(z^2)$ is inserted into Eq. (3.15), it yields

$$
-\operatorname*{Res}_{t=\tau_0} \bigl\{ e^{-kt}\varphi_0(t) \bigr\} = \hat{p}(k) \exp\bigl\{ O(kz^2) + O(z) \bigr\}, \qquad (3.16)
$$

where Eq. (1.3) defines $\hat{p}(k)$. In deriving Eq. (3.16), dominated convergence yields $\mathbb{E}(T^2|\mathcal{E}_y)z^2 = O\{\mathbb{E}(T^2; \mathcal{E}_y)z\} = o(z)$, even if $\lim_{y\to\infty} \mathbb{E}(T^2|\mathcal{E}_y) = \infty$.

In summary, Eqs. (3.8), (3.9), and (3.14) yield

$$\begin{aligned}
&\mathbb{P}\big\{N(k)=0\big\}-\mathbb{P}\big(K_0>k;\,\mathcal{E}_-^{(0)}\big)-\mathbb{P}\big(T_0>k;\,\mathcal{E}_y^{(0)}\big)\\
&\quad=\hat{p}(k)\exp\big\{O(kz^2)+O(z)\big\}\\
&\qquad+\lim_{n\to\infty}\Bigg[-\sum_{\{\tau\neq\tau_0\text{ in }\hat{R}(\pm 2r,\pm iw_n)\}}\operatorname*{Res}_{t=\tau}\big\{e^{-kt}\varphi_0(t)\big\}\\
&\qquad+(2\pi i)^{-1}\int\limits_{\hat{V}\uparrow(2r,\pm iw_n)}e^{-tk}\varphi_0(t)\,dt\Bigg].
\end{aligned}\tag{3.17}$$

Our present objective is therefore to bound the magnitude of the limit in Eq. (3.17). Unfortunately, the sum and the integral within the limit might not converge absolutely.

To obtain absolute convergence, introduce the following "smoothed" renewal-success process$(\bar{K}_i,\bar{T}_i(y),\mathbb{I}(\bar{\mathcal{E}}_i(y)))$. Let U be a uniform-$[0,1]$ random variable independent of the original renewal-success process, and let us couple [13] the smoothed process to the original process by defining $(\bar{K}_i,\bar{T}_i(y),\mathbb{I}(\bar{\mathcal{E}}_i(y))):=(K_i+U,T_i(y)+U,\mathbb{I}(\mathcal{E}_i(y)))$. Intuitively, the smoothed process is the original process, with a random uniform-$[0,1]$ delay. A related smoothing using a normal distribution appears in Eq. (7) of [19].

The following uses quantities with an over-stroke to refer to the smoothed process. Because $0\leqslant U=\bar{T}_i-T_i\leqslant 1$, the coupling inequality $\bar{N}(k)\leqslant N(k)\leqslant\bar{N}(k+1)$ follows from the definition of $N(k)$. Monotonicity gives $\mathbb{P}\{\bar{N}(k+1)=0\}\leqslant\mathbb{P}\{N(k)=0\}\leqslant\mathbb{P}\{\bar{N}(k)=0\}$. Because $\hat{p}(k+1)=\hat{p}(k)\exp\{O(z)\}$, $O\{(k+1)z^2\}=O(kz^2)+O(z^2)=O(kz^2)+O(z)$, and $O(e^{-r(k+1)})=O(e^{-rk})$, if Eq. (1.2) holds for the smoothed process, it also holds for the original process. The rest of the analysis therefore refers to the smoothed process.

The function $\mathbb{E}e^{tU}=\int_0^1 e^{tk}\,dk=t^{-1}(e^t-1)$ is entire, because its singularity at $t=0$ is removable. Note that for the smoothed process, $\mathbb{E}(e^{t\bar{K}_0};\,\mathcal{E}_-^{(0)})=\mathbb{E}e^{tU}\mathbb{E}(e^{tK_0};\,\mathcal{E}_-^{(0)})$ and $\mathbb{E}(e^{t\bar{T}_0};\,\mathcal{E}_y^{(0)})=\mathbb{E}e^{tU}\mathbb{E}(e^{tT_0};\,\mathcal{E}_y^{(0)})$, so the residues in Eq. (3.17) become absolutely summable,

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{\{\tau\neq\tau_0\text{ in }\hat{R}(\pm 2r,\pm iw_n)\}}\operatorname*{Res}_{t=\tau}\big\{e^{-kt}\bar{\varphi}_0(t)\big\}\\
&\quad=\lim_{n\to\infty}\sum_{\{\tau\neq\tau_0\text{ in }\hat{R}(\pm 2r,\pm iw_n)\}}\left|e^{-\tau k}\tau^{-2}(e^{\tau}-1)\frac{\mathbb{E}(e^{\tau K_0};\,\mathcal{E}_-^{(0)})\mathbb{E}(e^{\tau T};\,\mathcal{E}_y)}{\mathbb{E}(Ke^{\tau K};\,\mathcal{E}_-)}\right|\\
&\quad\leqslant\sum_{m\neq 0}\big|\tau_m^{-2}\big|\times\sup_{\tau\in S(0,3r)}\left\{\frac{|(e^{\tau}-1)\mathbb{E}(e^{\tau K_0};\,\mathcal{E}_-^{(0)})|}{|\mathbb{E}(Ke^{\tau K};\,\mathcal{E}_-)|}\big|e^{-\tau k}\mathbb{E}(e^{\tau T};\,\mathcal{E}_y)\big|\right\}\\
&\quad\leqslant\left\{\sum_{m\neq 0}\big|\tau_m^{-2}\big|\times\frac{|(e^{3r}-1)\mathbb{E}(e^{3rK_0};\,\mathcal{E}_-^{(0)})|}{\big(\frac{1}{8}\mathbb{E}K\big)}\right\}\sup_{0\leqslant u\leqslant 3r}\big\{e^{-uk}\mathbb{E}(e^{uT};\,\mathcal{E}_y)\big\}\\
&\quad=O\Big[\sup_{0\leqslant u\leqslant 3r}\big\{e^{-uk}\mathbb{E}(e^{uT};\,\mathcal{E}_y)\big\}\Big].
\end{aligned}\tag{3.18}$$

Lemma 2.6 guarantees that $\sum_{m\neq 0}|\tau_m^{-2}|<\infty$. Consequently, for the first inequality, the smoothed process enforces absolute summability, permitting rearrangement and factorization. (If $u:=\operatorname{Re} t<0$, then $|\mathbb{E}(e^{tK};\mathcal{E}_-)-1|\geqslant 1-\mathbb{E}(e^{uK};\mathcal{E}_-)\geqslant 1-\mathbb{E}e^{uK}>0$. Thus, no root τ of $\mathbb{E}(e^{\tau K};\mathcal{E}_-)-1=0$ satisfies $\operatorname{Re}\tau<0$, and $S\overline{(0,3r)}$ contains all the poles enclosed by $\hat{R}(\pm 2r,\pm i w_n)$.) For the second inequality, majorization provides a uniform bound on the absolute value of the numerator within $S\overline{(0,3r)}$, as in Eq. (3.10). In addition, Lemma 2.4 shows $|\mathbb{E}(Ke^{\tau K};\mathcal{E}_-)|\geqslant \frac{1}{8}\mathbb{E}K$. Equation (3.18) follows.

(Whereas [17] used Eq. (3.6), this paper favored Eq. (3.7), because the smoothed process complicates the analysis of the terms dropped in passing from Eq. (3.6) to Eq. (3.7).)

In Eq. (3.18), $f(u):=e^{-uk}\mathbb{E}(e^{uT};\mathcal{E}_y)$ satisfies $f''(u)=\mathbb{E}\{e^{u(T-k)}(T-k)^2;\mathcal{E}_y\}>0$. Thus, convexity shows that its supremum over $0\leqslant u\leqslant 3r$ occurs at $u=0$ or $u=3r$,

$$\begin{aligned}\sup_{0\leqslant u\leqslant 3r}\left\{e^{-uk}\mathbb{E}(e^{uT};\mathcal{E}_y)\right\}&=\max\left\{\mathbb{P}(\mathcal{E}_y),e^{-3rk}\mathbb{E}(e^{3rT};\mathcal{E}_y)\right\}\\&=O\left\{\mathbb{P}(\mathcal{E}_y)\right\}+O(e^{-3rk}).\end{aligned}\tag{3.19}$$

The integral in Eq. (3.17) is absolutely integrable for the smoothed process,

$$\begin{aligned}&\left|\int_{\hat{V}^{\uparrow}(2r,\pm i w_n)} e^{-kt}\bar{\varphi}_0(t)\,dt\right|\leqslant\int_{\hat{V}^{\uparrow}(2r,\pm i w_n)}\left|e^{-kt}\bar{\varphi}_0(t)\right||dt|\\&\quad=\int_{\hat{V}^{\uparrow}(2r,\pm i w_n)}|e^{-kt}||t^{-2}|\\&\qquad\times\left[|e^t-1|\frac{|\mathbb{E}(e^{tK_0};\mathcal{E}_-^{(0)})||1-\mathbb{E}(e^{tK};\mathcal{E}_-)-\mathbb{E}(e^{tT};\mathcal{E}_y)|}{|1-\mathbb{E}(e^{tK};\mathcal{E}_-)|}\right]|dt|\\&\quad\leqslant e^{-rk}\int_{-\infty}^{\infty}\frac{\sqrt{2}\,dv}{r^2+v^2}\\&\qquad\times\left[|e^{3r}-1|\frac{\mathbb{E}(e^{3rK_0};\mathcal{E}_-^{(0)})\{1+\mathbb{E}(e^{3rK};\mathcal{E}_-)+\mathbb{E}(e^{3rT};\mathcal{E}_y)\}}{\frac{1}{2}\varepsilon}\right]\\&\quad=O(e^{-rk}).\end{aligned}\tag{3.20}$$

Because $\hat{V}^{\uparrow}(2r,\pm i w_n)\subset S\overline{(r,3r)}$ (see Fig. 1), we have $r\leqslant\operatorname{Re} t\leqslant 3r$ in the integrand. Most of the justification of Eq. (3.20) is found in arguments following Eq. (3.10), with three additional points: (1) the extra factor $t^{-1}(e^t-1)$ occurs because of smoothing, (2) $|dt|\leqslant dv\sqrt{2}$, because the broken line segments composing $\hat{V}^{\uparrow}(2r,\pm i w_n)$ make an angle of 45° with the pure imaginary direction, and (3) $|\mathbb{E}(e^{tK};\mathcal{E}_-)-1|\geqslant\frac{1}{2}\varepsilon$ by Lemma 2.5.

Because $e^{-3rk}=O(e^{-rk})$, $e^{-\tilde{r}k}=O(e^{-rk})$, and $\mathbb{P}(\mathcal{E}_y)=O(z)$, Eqs. (3.8) and (3.17)–(3.20) prove Eq. (1.2) for $k>0$ a.e. (only "a.e.," because Eq. (3.9) is a Fourier inversion). The following "skeleton argument" completes the proof for all $k>0$.

Fix any real number $m>0$, e.g., $m=1$. Assume for every $k>m$, Eq. (1.2) holds for two numbers $k_\pm$, with $k-m\leqslant k_-\leqslant k\leqslant k_+\leqslant k+m$. By monotonicity, $\mathbb{P}\{N(k_+)=0\}\leqslant\mathbb{P}\{N(k)=0\}\leqslant\mathbb{P}\{N(k_-)=0\}$. Because $\hat{p}(k_\pm)=\hat{p}(k)\exp\{O(z)\}$, $k_\pm z^2=O(kz^2)+$

$O(z^2) = O(kz^2) + O(z)$, and $e^{-rk_\pm} = O(e^{-rk})$, Eq. (1.2) must hold for any $k > m$. It also holds for $0 < k \leqslant m$: just increase the constant of the final $O(e^{-rk})$ if necessary. The skeleton argument completes the proof of Eq. (1.2) in Theorem 1.1.

4. The finite-size correction in multiple processes

This section considers multiple renewal-success processes, deriving Eq. (1.6) for $\hat{\lambda}$ and the bound on $d_{TV}(N, Y_{\hat{\lambda}})$ in Eq. (1.7) of Theorem 1.2 by the Chen–Stein method. Following the plan in [17], it uses the analytic techniques in Section 3 to examine $\mathbb{E}N(k)$ and to bound $\mathbb{E}[N(k)\{N(k)-1\}]$. Like Section 3, it uses the lemmas of Section 2 freely, assigning the appropriate values to ε, r, and y implicitly.

In analogy to Eq. (3.6), the combinatorial formulas for the generating functions of $\mathbb{E}N(k)$ and $\mathbb{E}[N(k)\{N(k)-1\}]$ are

$$\varphi_1(1,t) = \int_0^\infty e^{tk}\mathbb{E}N(k)\,dk = -t^{-1}\left\{\frac{\mathbb{E}e^{tK_0}\mathbb{E}(e^{tT};\mathcal{E}_y)}{1-\mathbb{E}e^{tK}} + \mathbb{E}\big(e^{tT_0};\mathcal{E}_y^{(0)}\big)\right\} \tag{4.1}$$

and

$$\begin{aligned}\varphi_{11}(1,t) &= \int_0^\infty e^{tk}\mathbb{E}\big[N(k)\big\{N(k)-1\big\}\big]\,dk \\ &= -2t^{-1}\frac{\mathbb{E}(e^{tT};\mathcal{E}_y)}{1-\mathbb{E}e^{tK}}\left\{\frac{\mathbb{E}e^{tK_0}\mathbb{E}(e^{tK};\mathcal{E}_y)}{1-\mathbb{E}e^{tK}} + \mathbb{E}\big(e^{tK_0};\mathcal{E}_y^{(0)}\big)\right\},\end{aligned} \tag{4.2}$$

valid for $\operatorname{Re} t < 0$. Both Eqs. (4.1) and (4.2) indicate corrections to their counterparts in [17]. Their derivations are not given here, because they can be found in [17].

We now turn to proving Theorem 1.2. The same methods that applied to $\mathbb{P}\{N(k)=0\}$ in Eq. (3.9) also apply to $\mathbb{E}N(k)$ in Eq. (4.1), so the presentation is brief. For y large enough,

$$\begin{aligned}\mathbb{E}N(k) &= \lim_{n\to\infty}(2\pi i)^{-1}\int_{V^\uparrow(-c,\pm iw_n)} e^{-kt}\varphi_1(1,t)\,dt \\ &= \lim_{n\to\infty}\left[-\sum_{(\zeta)}\operatorname*{Res}_{t=\zeta}\big\{e^{-kt}\varphi_1(1,t)\big\} \right.\\ &\qquad \left. + (2\pi i)^{-1}\int_{\hat{V}^\uparrow(2r,\pm iw_n)} e^{-kt}\varphi_1(1,t)\,dt\right],\end{aligned} \tag{4.3}$$

where the summation is over all roots ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$ enclosed by $\hat{R}(\pm 2r, \pm iw_n)$. (Figure 1 in Section 3 defines $\hat{R}(\pm 2r, \pm iw_n)$, $V^\uparrow(-c, \pm iw_n)$, and $\hat{V}^\uparrow(2r, \pm iw_n)$.)

The residue calculation at $t = 0$ yields an estimate

$$\begin{aligned}\hat{N}(k) &:= -\operatorname*{Res}_{t=0}\bigl\{e^{-kt}\varphi_1(1,t)\bigr\}\\ &= \mathbb{P}\bigl(\mathcal{E}_y^{(0)}\bigr) + \frac{\mathbb{P}(\mathcal{E}_y)}{\mathbb{E}K}\left\{k - \mathbb{E}(T|\mathcal{E}_y) + \frac{1}{2}\frac{\mathbb{E}K^2}{\mathbb{E}K} - \mathbb{E}K_0\right\}\end{aligned} \tag{4.4}$$

for $\mathbb{E}N(k)$. Equation (4.4) lacks a continuity correction appearing in its analog in [17]. The arguments of Section 3 go through unchanged, except for the substitution of Lemmas 2.1–2.3 for Lemmas 2.4–2.6. The result is to show that

$$\mathbb{E}N(k) = \hat{N}(k) + O\bigl\{\mathbb{P}(\mathcal{E}_y)\bigr\} + O(e^{-rk}). \tag{4.5}$$

The analog of Eq. (4.5) in [17] lacks the $O\{\mathbb{P}(\mathcal{E}_y)\}$ term.

Similarly, for y large enough,

$$\begin{aligned}\mathbb{E}\bigl[N(k)\bigl\{N(k)-1\bigr\}\bigr] &= \lim_{n\to\infty}(2\pi i)^{-1}\int\limits_{V\uparrow(-c,\pm i w_n)} e^{-kt}\varphi_{11}(1,t)\,dt\\ &= \lim_{n\to\infty}\Biggl[-\sum_{(\zeta)}\operatorname*{Res}_{t=\zeta}\bigl\{e^{-kt}\varphi_{11}(1,t)\bigr\}\\ &\qquad + (2\pi i)^{-1}\int\limits_{\hat{V}\uparrow(2r,\pm i w_n)} e^{-kt}\varphi_{11}(1,t)\,dt\Biggr],\end{aligned} \tag{4.6}$$

where the summation is over all roots ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$ enclosed by $\hat{R}(\pm 2r, \pm i w_n)$.

A tedious residue calculation at $t = 0$ yields an estimate

$$\begin{aligned}-\operatorname*{Res}_{t=0}\bigl\{e^{-kt}\varphi_{11}(1,t)\bigr\} &= O\bigl[\bigl\{\mathbb{P}(\mathcal{E}_y)\bigr\}^2\bigl\{k^2 + \mathbb{E}(K^2|\mathcal{E}_y)\bigr\}\\ &\qquad + \mathbb{P}\bigl(\mathcal{E}_y^{(0)}\bigr)\mathbb{P}(\mathcal{E}_y)\bigl\{k + \mathbb{E}(T|\mathcal{E}_y) + \mathbb{E}\bigl(K_0|\mathcal{E}_y^{(0)}\bigr)\bigr\}\bigr],\end{aligned} \tag{4.7}$$

as in [17]. Finally, an estimate of the remaining residues and the integral yields

$$\mathbb{E}\bigl[N(k)\bigl\{N(k)-1\bigr\}\bigr] = -\operatorname*{Res}_{t=0}\bigl\{e^{-kt}\varphi_{11}(1,t)\bigr\} + O\bigl\{\mathbb{P}(\mathcal{E}_y)\bigr\} + O(e^{-rk}). \tag{4.8}$$

As for Eq. (4.5), the analog of Eq. (4.8) in [17] lacks the $O\{\mathbb{P}(\mathcal{E}_y)\}$ term.

Because of Fourier inversion, Eqs. (4.5) and (4.8) appear to hold only in a smoothed renewal-success process for $k > 0$ a.e. Because $\mathbb{E}N(k)$ and $\mathbb{E}[N(k)\{N(k)-1\}]$ are monotonic, however, the skeleton argument in Section 3 proves them for all $k > 0$.

To finish the proof of Eq. (1.7) in Theorem 1.2, let $\lambda := \sum_{j=1}^{A}\mathbb{E}N_j$. Note $\hat{\lambda} := \sum_{j=1}^{A}\hat{N}(k_j) + O\{A\mathbb{P}(\mathcal{E}_y)\}$, where $A\mathbb{P}(\mathcal{E}_y) = O(AV^{-1})$. In [17], the analog of Theorem 1.2 was based on analogs of Eqs. (4.5) and (4.8), to which this paper added the extra term $O\{\mathbb{P}(\mathcal{E}_y)\}$. The error bound for [17]'s Theorem 1.2 is essentially the sum of three quantities, $|\lambda - \hat{\lambda}|$ and two others called b_1 and b_2, which appear in a standard treatment of the Chen–Stein method [6]. The quantity $|\lambda - \hat{\lambda}|$ propagates the extra term $O\{\mathbb{P}(\mathcal{E}_y)\}$ in Eq. (4.5) for each of the A independent renewal-success processes, adding an extra term $O\{A\mathbb{P}(\mathcal{E}_y)\} = O(AV^{-1})$ to the right of Eq. (1.7). The estimate b_1 in [17] dominates the extra terms it receives from the $O\{\mathbb{P}(\mathcal{E}_y)\}$ term and is unchanged. Like $|\lambda - \hat{\lambda}|$, the estimate b_2 propagates its extra term $O\{\mathbb{P}(\mathcal{E}_y)\}$ from Eq. (4.8) for each of the A processes, producing a second (and therefore irrelevant) term $O\{A\mathbb{P}(\mathcal{E}_y)\} = O(AV^{-1})$ to the right of Eq. (1.7). The demonstration of Eq. (1.7) in Theorem 1.2 closely follows [17]'s demonstration of its Theorem 1.2. The reader should consult [17] for further details.

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